DIFFERENTIAL EQUATION

A. DEFINITION

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION**.

There are two types of differential equation :

(i) Ordinary Differential Equation : A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$

(ii) Partial Differential Equation : A differential equation is said to be partial, if there are two or more independent variables, e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation.

We are concerned with ordinary differential equations only.

(a) Solution (primitive) of differential equation : Finding the unknown function which satisfies given differential equation is called SOLVING OR INTEGRATING the differential equation. The solution of the differential equation is also called its PRIMITIVE, because the differential equation can be regarded as a relation derived from it.

(b) Order of differential equation : The order of a differential equation is the order of the highest differential coefficient occurring in it.

(c) Degree of differential equation : The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form which is free from radicals and fractions so far as derivatives are concerned, thus the differential equation $f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots = 0$

is order m and degree p.

Note that : In the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't apply.

Ex.1 Find the order and degree of the following differential equation :

(i) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$ **(ii)** $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$ **(iii)** $\frac{dy}{dx} = \sqrt{3x+5}$

Sol. **(i)** The given differential equation can be re-written as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$

Hence order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is obviously 1 and degree 1.

Ex.2 The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are

Sol. Clearly order is 2 and degree is 2. (from the definition of order and degree of differential equation).

B. FORMATION OF A DIFFERENTIAL EQUATION

If an equation with independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

The eliminant is the required differential equation.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

Ex.3 Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Sol. Equation of parabola whose axes is parallel to x-axis and having latusrectum 'a' is $(y - \beta)^2 = a\,(x - \alpha)$

Differentiating both sides, we get $2(y - \beta)\frac{dy}{dx} = a \Rightarrow 2(y - \beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow a.\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$

Ex.4 Find the differential equation whose solution represents the family : $c\,(y + c)^2 = x^3$

Sol. $c\,(y + c)^2 = x^3$ ...(i) Differentiating, we get, $c.[2(y + c)]\frac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have $\frac{2x^3}{(y+c)^2}(y+c)\frac{dy}{dx} = 3x^2 \Rightarrow \frac{2x^3}{y+c}\frac{dy}{dx} = 3x^2$

i.e. $\frac{2x}{y+c}\frac{dy}{dx} = 3 \Rightarrow \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c$ Hence $c = \frac{2x}{3}\left[\frac{dy}{dx}\right] - y$

Substituting value of c in equation (i), we get $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$,

Which is the required differential equation.

C. GENERAL AND PARTICULAR SOLUTIONS

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE). A solution obtainable from the general solution by giving particular values to the constants is called a PARTICULAR SOLUTION.

Note that : the general solution of a differential equation of the n^{th} order contains 'n' and only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A,B in the equation $y = A e^{x+B}$ are not independent since the equation can be written as $y = Ae^B \cdot e^x = C e^x$ .Similarly the solution $y = A \sin x + B \cos (x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

D. ELEMENTARY TYPES OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

(1) VARIABLES SEPARABLE :

Type – 1 : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable separable type. A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant. Consider the example $(dy/dx) = e^{x-y} + x^2.e^{-y}$.

Ex.5 Solve the differential equation $xy \frac{dy}{dx} = \frac{1+y^2}{1+x^2} f\,(1 + x + x^2)$.

Sol. Differential equation can be rewritten as $xy \frac{dy}{dx} = (1 + y^2)\left(1+\frac{x}{1+x^2}\right) \Rightarrow \frac{y}{1+y^2}dy = \frac{1}{x}+\frac{1}{1+x^2}dx$

Integrating, we get $\frac{1}{2}\ell n\,(1 + y^2) = \ell n\, x + \tan^{-1} x + \ell n\, c \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1} x}$.

Ex.6 Solve the differential equation $(x^3 - y^2x^3) \frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Sol. The given equation $(x^3 - y^2x^3) \frac{dy}{dx} + y^3 + x^2y^3 = 0$

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \Rightarrow \int\left(\frac{1}{y^3}-\frac{1}{y}\right)dy + \int\left(\frac{1}{x^3}+\frac{1}{x}\right) dx = 0 \Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right) + c$$

TYPE – 2 : $\frac{dy}{dx} = f(ax + by + c),\ b \neq 0$. To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x + y)^2 \frac{dy}{dx} = a^2$

TYPE – 3 : $\frac{dy}{dx} = \frac{a_1x + b_1y_1 + c_1}{a_2x + b_2y + c_2}$, $b_1 + a_2 = 0$. To solve this, simple cross multiplication and substituting d (xy) for xdy + ydx and integrating term by term yields the result easily.

Consider the examples $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}; \frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$ and $\frac{dy}{dx} = \frac{2x-y+1}{6x-5y+4}$.

TYPE – 4 : Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.

If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$ **(ii)** $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ **(iii)** $xdy - ydx = r^2\, d\theta$

If $x = r\sec\theta$ and $y = r\tan\theta$ then $x\,dx - y\,dy = rdr$ and $x\,dy - ydx = r^2\sec\theta\, d\theta$.

Ex.7 Solve $\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$

Sol. $\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$ Putting $x + y = t$, we get, $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore $\frac{dt}{dx} - 1 = \cos t - \sin t \Rightarrow \frac{dt}{1 + \cos t - \sin t} = dx \Rightarrow \frac{\sec^2\frac{t}{2}dt}{2\left(1 - \tan\frac{t}{2}\right)} = dx$

Integrating, we get, $-\ell n\left|1 - \tan\frac{x+y}{2}\right| = x + c$.

(2) HOMOGENEOUS EQUATIONS :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x & y and of the same degree, is called HOMOGENEOUS. This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ and is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider the example $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$.

Ex.8 The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is

Sol. Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ $\Rightarrow$ $\cos y\,\frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$, (put $\sin y = t$)

$\Rightarrow \frac{dt}{dx} = \frac{t + x}{2t - x}$, (put $t = vx$) $\Rightarrow \frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v + 1}{2v - 1}$

$\therefore x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v + 1 - 2v^2 + v}{2v - 1}$ or $\frac{2v^2 - v}{-2v^2 + 2v + 1}dy = \frac{dx}{x}$ on solving we get $\sin^2 y = x\sin y + \frac{x^2}{2} + c$

Ex.9 A solution of the equation $x\frac{dy}{dx} = y(\log y - \log x + 1)$ is

Sol. Putting $y = vx$ in the given equation, we have $v + x\frac{dv}{dx} = v(\log v + 1) \Rightarrow x\frac{dv}{dx} = v\log v$

$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x} \Rightarrow \log|\log v| = \log|x| + \log c \Rightarrow \log v = cx \Rightarrow y = xe^{cx}$

Equations reducible to the homogeneous form :

If $\frac{dy}{dx} = \frac{a_1x + b_1y_1 + c_1}{a_2x + b_2y + c_2}$; where $a_1b_2 - a_2b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ then the substitution x = u + h, y = v + k transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in (b). If

(i) $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable and

(ii) In an equation of the form : yf(xy)dx + xg(xy)dy = 0 the variables can be separated by the substitution xy = v.

Important note :

(i) The function f(x, y) is said to be a homogeneous function of degree n if for any real number t($\neq$0), we have f(tx, ty) = t^n f(x, y). For e.g. f(x, y) = $ax^{2/3} + hx^{1/3} . y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

(ii) A differential equation of the form $\frac{dy}{dx}$ = f(x , y) is homogeneous if f(x, y) is a homogeneous function of degree zero i.e. f(tx, ty) = t^o f(x, y) = f(x, y). The function f does not depend on x and y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

Ex.10 Solve $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Sol. Put x = X + h, y = Y + k. We have $\frac{dY}{dX} = \frac{X + 2Y + (h+2k+3)}{2X + 3Y + (2h + 3k + 4)}$

To determine h and k, we write h + 2k + 3 = 0, 2h + 3k + 4 = 0 $\Rightarrow$ h = 1, k = –2 so that $\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$

Putting Y= VX, we get V + X $\frac{dV}{dX} = \frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1}dV = -\frac{dX}{X}$

$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X} \Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1) = (-\log X + c)$

$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y - X) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y + X) = A$ where A is another constant and X = x – 1, Y= y+ 2.

Ex.11 Solve the differential equation $(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$.

Sol. The equation is homogeneous of degree 0. Put x = vy, dx = v dy + y dv,

Then $(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0 \Rightarrow (v + 2e^v)\,dy + y(1 + 2e^v)\,dv = 0$

$\frac{dy}{y} + \frac{1+2e^v}{v+2e^v}dv = 0$. Integrating and replacing v by x/y

$\ell n\, y + \ell n\,(v + 2e^v) = \ell nC$ and $x + 2ye^{x/y} = c$

(3) LINEAR DIFFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together. The nth order linear differential equation is of the form ; $a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + a_n(x) \cdot y = \phi(x)$, where $a_0(x), a_1(x) \ldots a_n(x)$ are called the coefficients of the differential equation.

Note that : A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

Ex.12 Which of the following equation is linear ?

(A) $\frac{dy}{dx} + xy^2 = 1$ (B) $x^2\frac{dy}{dx} + y = e^x$ (C) $\frac{dy}{dx} + 3y = xy^2$ (D) $x\frac{dy}{dx} + y^2 = \sin x$

Sol. Clearly answer is (B)

Ex.13 Which of the following equation is non-linear ?

(A) $\frac{dy}{dx} = \cos x$ (B) $\frac{d^2y}{dx^2} + y = 0$ (C) $dx + dy = 0$ (D) $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

Sol. Clearly answer is (D)

(a) Linear differential equations of first order : The most general form of a linear differential equations of the first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int P dx}$.

Note :

(i) The factor $e^{\int P dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of y and the I.F.

(iii) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ; $(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

Ex.14 Solve $(1 + y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0$.

Sol. Differential equation can be rewritten as $(1 + y^2)\dfrac{dx}{dy} + x = e^{\tan^{-1}y}$ or $\dfrac{dx}{dy} + \dfrac{1}{1+y^2}\cdot x = \dfrac{e^{\tan^{-1}y}}{1+y^2}$ ....(i)

I. F = $e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$ so solution is $xe^{\tan^{-1}y} = \int\dfrac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2}dy$

Let $e^{\tan^{-1}y} = t \Rightarrow \dfrac{e^{\tan^{-1}y}}{1+y^2}dy = dt \Rightarrow xe^{\tan^{-1}y} = \int t\,dt$ [Putting $e^{\tan^{-1}y} = t$]

or $xe^{\tan^{-1}y} = \dfrac{t^2}{2} + \dfrac{c}{2} \Rightarrow 2x\,e^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$.

Ex.15 The solution of differential equation $(x^2 - 1)\dfrac{dy}{dx} + 2xy = \dfrac{1}{x^2-1}$ is

Sol. The given differential equation is $(x^2 - 1)\dfrac{dy}{dx} + 2\,xy = \dfrac{1}{x^2-1} \Rightarrow \dfrac{dy}{dx} + \dfrac{2x}{x^2-1}y = \dfrac{1}{(x^2-1)^2}$

This is linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{2x}{x^2-1}$ and $Q = \dfrac{1}{(x^2-1)^2}$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int 2x/(x^2-1)\,dx} = e^{\log(x^2-1)} = (x^2-1)$

multiplying both sides of (i) by I.F. = $(x^2 - 1)$, we get $(x^2 - 1)\dfrac{dy}{dx} + 2\,xy = \dfrac{1}{x^2-1}$

integrating both sides we get $y(x^2 - 1) = \int\dfrac{1}{x^2-1}dx + C$ [Using : y (I.F.) = $\int Q.(I.F.)\,dx + C$]

$\Rightarrow y(x^2 - 1) = \dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right| + C$. This is the required solution.

(b) EQUATIONS REDUCIBLE TO LINEAR FORM : The equation $\frac{dy}{dx}$ + Py = Q.y^n where P & Q are function, of x, is reducible to the linear form by dividing it by y^n & then substituting y^{-n+1} = Z. Its solution can be obtained as in **(a)** . Consider the example $(x^3y^2 + xy)dx = dy$.

The equation $\frac{dy}{dx}$ + Py = Qy^n is called **BERNOULI'S EQUATION**.

Ex.16 Solve the differential equation $x\frac{dy}{dx} + y = x^3y^6$.

Sol. The given differential equation can be written as $\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$

Putting $y^{-5} = v$ so that $-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$ or $y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}$ we get

$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2$ ....(i)

This is the standard form of the linear differential equation having integrating factor

I.F = $e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$

Multiplying both sides of (i) by I.F. and integrating w.r.t r

We get $v.\frac{1}{x^5} = \int -5x^2.\frac{1}{x^5}dx \Rightarrow \frac{v}{x^5} = \frac{5}{2}x^{-2} + c \Rightarrow y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$ which is the required solution.

Ex.17 Find the solution of differential equation $\frac{dy}{dx} - y\tan x = -y^2\sec x$.

Sol. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$ (Put $\frac{1}{y} = v$; $\frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$) $\therefore \frac{-dv}{dx} - v\tan x = -\sec x$

$\frac{dv}{dx} + v\tan x = \sec x$, Here P = tan x, Q = sec x

I.F. = $e^{\int \tan x dx} = \sec x$; $v\sec x = \int \sec^2 x\,dx + c$ Hence the solution is $y^{-1}\sec x = \tan x + c$

SOME IMPORTANT DIFFERENTIALS MUST BE REMEMBERED :

(i) $xdy + y\,dx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d(\ell nxy)$

(v) $\dfrac{dx + dy}{x + y} = d(\ell n(x + y))$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

Ex.18 Solve $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$.

Sol. The given equation can be written as $\dfrac{xdx + ydy}{(x^2 + y^2)^2} = \dfrac{ydx - xdy}{y^2} \cdot \dfrac{y^2}{x^2} \Rightarrow \int \dfrac{d(x^2 + y^2)}{(x^2 + y^2)^2} = 2\int \dfrac{1}{x^2/y^2}\, d\left(\dfrac{x}{y}\right)$

Integrating both sides we get $-\dfrac{1}{(x^2 + y^2)} = -\dfrac{1}{\frac{x}{y}} + c \Rightarrow \dfrac{y}{x} - \dfrac{1}{(x^2 + y^2)} = c.$

Ex.19 Solve $\dfrac{y + \sin x \cos^2(xy)}{\cos^2(xy)}dx + \left(\dfrac{x}{\cos^2(xy)} + \sin y\right)dy = 0.$

Sol. The given differential equation can be written as ; $\dfrac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$

$\Rightarrow \sec^2(xy)\,d(xy) + \sin x\,dx + \sin y\,dy = 0$

$d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0 \quad \Rightarrow \quad \tan(xy) - \cos x - \cos y = c.$

E. TRAJECTORIES

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

A curve making at each of its points a right angle with the curve of the family passing through that point is called an orthogonal trajectory of that family.

ORTHOGONAL TRAJECTORIES :

We set up the differential equation of the given family of curves. Let it be of the form F(x, y, y') = 0

The differential equation of the orthogonal trajectories is of the form $F\left(x, y, \frac{-1}{y'}\right) = 0$

The general integral of this equation $\phi_1(x, y, C) = 0$ gives the family of orthogonal trajectories.

Ex.20 Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Sol. Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, we get $2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$

or $2x + (4y - 1)\frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$

$\Rightarrow \int\frac{dy}{4y-1} = \int\frac{dx}{2x} \Rightarrow \frac{1}{4}\ell n\,(4y - 1) = \frac{1}{2}\ell n\, x + \frac{1}{2}\ell n\, a$, where a is any constant.

$\Rightarrow \ell n(4y - 1) = 2\,\ell n\, x + 2\,\ell n\, a$ or , $4y - 1 = a^2x^2$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory,

which is of the form $y = cx^2 + k$ where $c = \frac{a^2}{4}$, $k = \frac{1}{4}$

Ex.21 Solve (y log x – 1) ydx = xdy.

Sol. The given differential equation can be written as $x\frac{dy}{dx} + y = y^2 \log x$ ....(i)

Divide by xy^2 . Hence $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$

Let $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ so that $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x$ ....(ii)

(ii) is the standard linear differential equation with $P = -\frac{1}{x}$, $Q = -\frac{1}{x}\log x$

I.F. = $e^{\int p dx} = e^{\int -1/x\, dx} = 1/x$

The solution is given by $v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x}\log x\right)dx = -\int \frac{\log x}{x^2}dx = \frac{\log x}{x} - \int \frac{1}{x}.\frac{1}{x}dx = \frac{\log x}{x} + \frac{1}{x} + c$

$\Rightarrow$ v = 1 + log x + cx = log ex + cx or $\frac{1}{y}$ = log ex + cx or y (log ex + cx) = 1.

Ex.22 The solution of $e^{x\frac{(y^2-1)}{y}}$ $\{xy^2\,dy + y^3\,dx\} + \{ydx - xdy\} = 0$, is

Sol. Hence $e^{x\frac{(y^2-1)}{y}} . y^2 \{xdy + ydx\} + \{ydx - xdy\} = 0 \Rightarrow e^{xy} . y^2. \{xdy + ydx\} + e^{x/y} \{ydx - xdy\} = 0$

$\Rightarrow$ $e^{xy} . \{xdy + ydx\} + e^{x/y}\frac{\{ydx - xdy\}}{y^2} = 0$ $\Rightarrow$ $e^{xy} . d(xy) + e^{x/y} . d\left(\frac{x}{y}\right) = 0$

or $d(e^{xy}) + d(e^{x/y}) = 0$, Integrating both sides we get $e^{xy} + e^{x/y} + c = 0$.

Ex.23 For a certain curve y = f(x) satisfying $\frac{d^2y}{dx^2} = 6x - 4$ f(x) has a local minimum value 5 when x = 1. Find the equation of the curve and also the global maximum and global minimum values of f(x) given that

$0 \le x \le 2$.

Sol. Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When x = 1, $\frac{dy}{dx} = 0$, so that A = 1. Hence $\frac{dy}{dx} = 3x^2 - 4x + 1$ ....(i)

Integrating, we get $y = x^3 - 2x^2 + x + B$

When x = 1, y = 5, so that B = 5. Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points x = 1/3, x = 1.

At the critical point $x = \frac{1}{3}, \frac{d^2y}{dx^2}$ is negative. Therefore at x = 1/3, y has a local maximum.

At $x = 1, \frac{d^2y}{dx^2}$ is positive. Therefore at x = 1, y has a local minimum

Also $f(1) = 5, f\left(\frac{1}{3}\right) = \frac{157}{27}$. $f(0) = 5, f(2) = 7$

Hence the global maximum value = 7, and the global minimum value = 5.

Ex.24 Determine the equation of the curve passing through the origin, in the form y = f(x), which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$.

Sol. We have $\frac{dy}{dx} = \sin(10x + 6y)$. ....(i)

Let t = 10x + 6y $\Rightarrow$ $\frac{dt}{dx} = 10 + 6\frac{dy}{dx} = 10 + 6 \sin t$ $\Rightarrow$ $\frac{dt}{5 + 3 \sin t} = 2 \quad dx \Rightarrow \int \frac{dt}{5 + 3\sin t} = 2x + C$ .....(ii)

Putting $\tan\frac{t}{2} = z$, we get $\int \frac{2dz}{(1 + z^2)\left[\frac{6z}{1 + z^2} + 5\right]} = 2x + C$ from (ii)

$$\Rightarrow \int \frac{dz}{5z^2+6z+5} = x + \frac{C}{2} \Rightarrow \frac{1}{5}\int \frac{dz}{\left[z+\frac{3}{5}\right]^2 + \left[\frac{4}{5}\right]^2} = x + \frac{C}{2}$$

$$\Rightarrow \frac{1}{4}\tan^{-1}\frac{5z+3}{4} = x + \frac{C}{2} \Rightarrow \tan^{-1}\left[\frac{5\tan\frac{10x+6y}{2}+3}{4}\right] = 4x + 2C$$

Since the curve passes through (0, 0) $\Rightarrow$ 2C = $\tan^{-1}$ (3/4)

Thus $5\tan(5x+3y)+3 = 4\tan[4x+(\tan^{-1}3/4)]$

Ex.25 The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 minutes, when will the temperature be 295K.

Sol. Let T be the temperature of the substance at a time t. Then $-\frac{dT}{dt} \propto (T-290) \Rightarrow \frac{dT}{dt} = -k(T-290)$

(negative sign. because $\frac{dT}{dt}$ is rate of cooling) $\Rightarrow \int \frac{dT}{T-290} = -k\int dt$ ....(i)

Integrating the L.H.S. between the limits, T = 330 to T = 370 and the RHS between the limits

t = 0 to t = 10, we get $\int_{370}^{330} \frac{dT}{T-290} = -k\int_0^{10} dt \Rightarrow \log(T-290)\Big|_{370}^{330} = -kt\Big|_0^{10}$

$\Rightarrow \log 40 - \log 80 = -k.10 \Rightarrow \log 2 = 10k$ $\Rightarrow k = \frac{\log 2}{10}$ ....(ii)

Now, integrating (i) between T = 370 and T = 295 and t = 0 and t = t, we get

$$\int_{370}^{295} \frac{dT}{T-290} = -k\int_0^{t} dt \Rightarrow \log(T-290)\,|_{370}^{295} = -kt \Rightarrow \log 5 - \log 80 = -kt \Rightarrow -\log 16 = -kt \Rightarrow t = \frac{\log 16}{k}$$

Hence, from (ii), $t = \frac{\log 16}{\log 2}10 = 40\,\text{minutes}$ i.e. after 40 minutes.